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Leveled Isogeny Problems with Hints

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Outline

1. Introduction
2. Attacks based on Coppersmith method
3. Combinatorial attack

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Isogeny Problem with Level Structure

- *SIDH attacks:* given $P, Q, \varphi(P), \varphi(Q)$ and $\deg(\varphi)$ can *recover* φ
- countermeasures: publish $\alpha P, \beta Q$ or hide the degree
- *isogeny problem with level structure* [EC:DeFFouPan24]

Leveled Isogeny Problem with Hints

- more generally: fix a subgroup $\Gamma \leq \mathrm{GL}_2(N)$
- publish $M \cdot (\varphi(P), \varphi(Q))^T$ for $M \in \Gamma$
- degree potentially unknown
- hints on $M \rightsquigarrow$ *leveled isogeny problem with hints*

Deriving the equation

- *Weil pairing:* given points (P, Q) of order N and $(\alpha\varphi(P), \beta\varphi(Q))$ it holds

$$e_N(\alpha\varphi(P), \beta\varphi(Q)) = e_N(P, Q)^{\deg(\varphi)\alpha\beta}$$

- e_N is an N-th root of unity (N generally smooth)
- we get an equation

$$\deg(\varphi) \cdot \det(M) - r \equiv 0 \mod N$$

for some known r

Coppersmith Method

Theorem (Coppersmith)

Given a modulus N, a univariate monic polynomial $f(x)$ of degree δ and a bound $X \in \mathbb{N}$, if

$$X < N^{1/\delta}$$

we can compute all $r < X$ such that $f(r) \equiv 0 \bmod N$ in time polynomial in $\log N$ and δ.

Coppersmith Method

- find small roots
- extensive cryptanalytic applications
- can be extended to multivariate polynomials
- requires more setup $\rightsquigarrow$ *automated Coppersmith* [AC:MeeNow23]

Notation

Definition

Let $x \in \mathbb{N}$ be an n-bit integer and $x[i]$ its i-th bit. A k-bit hint for x is a set $\mathbb{H}_{\mathcal{J}}(x) := \{x[j] : j \in \mathcal{J}\}$ where $\#\mathcal{J} = k$.

- most significant bits (MSB): $\mathbb{M}_k(x) := \mathbb{H}(x)$ with $\mathcal{J} = \{1, ..., k\}$.
- least significant bits (LSB): $\mathbb{L}_k(x) := \mathbb{H}(x)$ with $\mathcal{J} = \{n - k + 1, ..., n\}$
- random bits: $\mathbb{R}_k(x) := \mathbb{H}(x)$ with random $\mathcal{J}$ of size k
- continuous bits: $\mathbb{C}_k(x) := \mathbb{H}(x)$ with $\mathcal{J} = \{s + i \bmod n, 0 \leq i < k\}$

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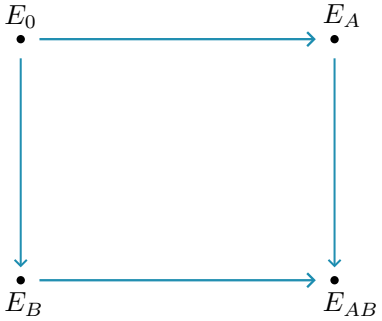
Scalar matrices

$$M = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$$

Applications:

- MSIDH (known degree)
- POKE (4D variant, unknown degree)



Scalar matrices - MSIDH

- public key: $(E', [\alpha]\varphi(P), [\alpha]\varphi(Q))$
- equation:

$$\alpha^2 - 1 \equiv 0 \bmod N$$

Theorem

There exists an efficient adversary agains LIPH with M a scalar matrix given $\mathbb{M}_k(M) = \mathbb{M}_k(\alpha)$ with $k = \lceil (\log N)/2 \rceil$.

Scalar matrices - POKE (4D variant)

- public key: $(E', [\alpha]\varphi(P), [\alpha]\varphi(Q))$
- unknown degree q
- equation:

$$\alpha^2 q - r \equiv 0 \bmod N$$

Theorem

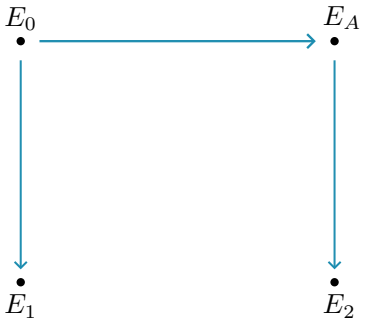
Let $q < N^{6/17}$. There exists an efficient adversary agains LIPH with M a scalar matrix given $\mathbb{M}_k(M) = \mathbb{M}_k(\alpha)$ with $k = \lceil (29 \cdot \log N)/34 \rceil$.

Diagonal matrices

$$M = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

Applications:

- FESTA



Diagonal matrices - FESTA

- public key: $(E', [\alpha]\varphi(P), [\beta]\varphi(Q))$
- equation:

$$\alpha\beta \equiv 1 \bmod 2^b$$

Theorem

There exists an efficient adversary against LIPH with M a diagonal matrix given $\mathbb{M}_{2k}(M) = (\mathbb{M}_k(\alpha), \mathbb{M}_k(\beta))$ with $k = \lceil 2b/3 \rceil$.

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Combinatorial attack - FESTA

- same equation:

$$\alpha\beta \equiv 1 \bmod 2^b$$

- random hints $\mathbb{R}_{b/2}(\alpha), \mathbb{R}_{b/2}(\beta) \rightsquigarrow$ *binary expansion with holes*
- exploit $\alpha\beta \equiv 1 \bmod 2^k$ for all $k \leq b$

Early pruning

$$
\begin{array}{r}
00{*}1 \\
1{**}1 \\
\hline
= \ 1 \mod 2^0
\end{array}
$$

$$Q_0 = \{(1, 1)\}$$

Early pruning

$$
\begin{array}{r}
00{*}1 \\
1{*}{*}1 \\
\hline
= \ 1 \mod 2^1
\end{array}
$$

$$Q_1 = \{(11, 11), (01, 01)\}$$

Early pruning

$$
\begin{array}{r}
00*1 \\
1**1 \\
\hline
= \ 1 \mod 2^2
\end{array}
$$

$$Q_2 = \{(011, 011), (001, 001)\}$$

Early pruning

$$\begin{array}{r} 00{*}1 \\ 1{**}1 \\ \hline = \ 1 \mod 2^3 \end{array}$$

$$Q_3 = \{(0011, 1011)\}$$

Statistical analysis

- Q_k is the list of all solutions at step k
- total complexity: $O(b \cdot \max_k |Q_k|)$
- goal: compute $\max_k \mathbb{E}[|Q_k|] = \mathbb{E}[|Q_b|]$

A dice game

- define $J_k = \log_2(|Q_k|)$
- start with 0 coins, and throw a 4-sided dice (A/B/C/D)
- if A or B, keep coins unchanged
- if C, gain one coin
- if D and you have at least one coin, lose a coin
- J_k is the number of coins after k turns

A dice game

- outcomes after 0 turns: $\{0\}$
- after 1 turn: $\{0, 0, 0, 1\}$
- after 2 turns: $\{0^{10}, 1^5, 2\}$
- ...

Statistical analysis

- $\mathbb{E}[J_k] = O(\sqrt{k})$
- unfortunately, $2^{\mathbb{E}[J_k]} \leq \mathbb{E}[2^{J_k}] = \mathbb{E}[|Q_k|]$ *(Jensen's inequality)*
- indeed $\mathbb{E}[2^{J_k}] \approx 1.125^k$
- on the other hand $\mathbb{P}(J_k < \sqrt{k}) \approx 0.84$
- very practical on concrete instances

Conclusion

- MSIDH: 50% of MSB
- POKE 4D: 86% of MSB
- FESTA: 67% of MSB
- FESTA (combinatorial): 50% random bits, $O(\sqrt{b})$ with 84% probability

Thank you for your attention.
Questions?